

A natural representation model for symmetric groups

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Abstract

For any positive integer N , we describe a natural complex representation of the symmetric group Σ_N on the vector space spanned by its involutions that contains each irreducible representation exactly once.

1 Introduction

Given a finite group G , it is interesting to construct a ‘natural representation’ of G that contains each of its irreducible representations exactly once. Such a representation is called a Gelfand model for G in [Stn] and constructed for the general linear groups over finite fields in [Kly]. In [GdoRjo], such a model is described for the family of symmetric groups where the representation space is obtained as the zeroes of certain differential operators in the Weyl algebra. In this note, we construct another very simple model for this family. It would be interesting to explicitly compare the two models and give a construction of either model for, say, the finite Weyl groups.

Throughout this paper we adopt the following notations and conventions. Let N be a fixed positive integer and $G = \Sigma_N$ be the symmetric group of permutations of the set $[N] := \{1, 2, \dots, N\}$. The identity element of G will be denoted 1_G . For $0 \leq j \leq \lfloor N/2 \rfloor$, let X_j be the conjugacy class of involutions - elements with square 1_G - of G that are a product of j disjoint transpositions. If $\tau \in X_j$ we will speak of τ as having length j and write $\ell(\tau) = j$. Let $X = \coprod_{j=0}^{\lfloor N/2 \rfloor} X_j$ be the set of all involutions of G . All our vector spaces and representations will be defined over the field of complex numbers. Let V_j be the vector space with basis X_j .

Each V_j admits a representation π_j of G defined on the basis X_j as follows: take $\tau \in X_j$ and write it as $(p_1 p_2)(p_3 p_4) \cdots (p_{2j-1} p_{2j})$ where $p_{2k-1} <$

p_{2k} for each $k = 1, 2, \dots, j$. For $\sigma \in G$, define:

$$\pi_j(\sigma)(\tau) = S(\sigma, \tau) (\sigma(p_1)\sigma(p_2))(\sigma(p_3)\sigma(p_4)) \cdots (\sigma(p_{2j-1})\sigma(p_{2j}))$$

where $S(\sigma, \tau) \in \{\pm 1\}$ is defined to be $(-1)^{\#\{k: \sigma(p_{2k-1}) > \sigma(p_{2k})\}}$ - where $\#$ stands for ‘cardinality of’. Equivalently, in matrix terms, we have that $\pi_j(\sigma)_{\tau}^{\tau'} = S(\sigma, \tau) \delta_{\tau', \sigma \tau \sigma^{-1}}$.

In order to see that π_j is indeed a representation of G , we note that an alternative description is as follows. Let V denote the ‘natural’ representation space of Σ_N with a basis v_1, \dots, v_N on which Σ_N acts as $\sigma(v_k) = v_{\sigma(k)}$. Consider the representation $\bigwedge^2 V$ which has basis $v_{ij} := v_i \wedge v_j$ for $1 \leq i < j \leq N$. Identifying $Sym(\bigwedge^2 V)$ with the polynomial ring in the variables v_{ij} , it is easy to see that V_j is isomorphic to the j^{th} graded component of the G -module $Sym(\bigwedge^2 V)/I$ where I is the monomial ideal generated by all $v_{ij}v_{kl}$ where $\{i, j\} \cap \{k, l\} \neq \emptyset$. This description has the virtue of providing a G -module algebra structure on $A = \bigoplus_{j=1}^{\lfloor N/2 \rfloor} V_j$.

Observe that the fact that π_j is a representation of G implies that the function S satisfies the following basic equation:

$$S(\sigma' \sigma, \tau) = S(\sigma', \sigma \tau \sigma^{-1}) S(\sigma, \tau), \quad (1)$$

for any $\sigma, \sigma' \in G$ and $\tau \in X$ and is, in particular, a character of the centraliser of τ .

We may now state our main theorem.

Theorem 1 *The G -module $A = \bigoplus_{j=1}^{\lfloor N/2 \rfloor} V_j$ contains every irreducible representation of G exactly once.*

The strategy of proof is very simple. We first show that the representations π_j are mutually disjoint in the sense that they do not contain any non-zero isomorphic G -subrepresentations. Then we show that the G -endomorphism ring of any V_j is abelian thus establishing that it is multiplicity free. Finally we show that the dimension of $End_G(A)$ is equal to the number of partitions of N showing that every irreducible representation of G features in A .

Much of the calculations are based on analysing matrix entries of a G -map from V_j to V_k with respect to the distinguished bases X_j and X_k . So let $T : V_j \rightarrow V_k$ be a G -map. We will identify T with its matrix $((T_{\tau}^{\kappa}))$ for $\tau \in X_j$ and $\kappa \in X_k$. Thus $T(\tau) = \sum_{\kappa} T_{\tau}^{\kappa} \kappa$. That T is a G -map is seen to translate into the equations:

$$S(\sigma, \kappa) T_{\tau}^{\kappa} = S(\sigma, \tau) T_{\sigma \tau \sigma^{-1}}^{\sigma \kappa \sigma^{-1}} \quad (2)$$

for all $\tau \in X_j$ and $\kappa \in X_k$.

2 Involutions in the symmetric group

To proceed, we will need to analyse the structure of the set $X \times X$ as a G -set under the simultaneous conjugation action. Given an element $(\tau, \kappa) \in X \times X$, we define its characteristic partition, denoted $P(\tau, \kappa)$, to be the finest partition, say $\{H_1, H_2, \dots, H_t\}$, of $[N]$ such that both τ and κ are in the subgroup $\Sigma_{H_1} \times \dots \times \Sigma_{H_t}$ (with the obvious meaning) of Σ_N . There are unique expressions $\tau = \tau_1 \tau_2 \dots \tau_t$ and $\kappa = \kappa_1 \kappa_2 \dots \kappa_t$ where $\tau_i, \kappa_i \in \Sigma_{H_i}$ for $i = 1, 2, \dots, t$. This also specifies a partition $p(\tau, \kappa)$ of the integer N as $\sum_{i=1}^t |H_i|$ which we will refer to as the numerical characteristic partition of (τ, κ) . Clearly $p(\tau, \kappa)$ is an invariant of the orbit of (τ, κ) .

Lemma 2 *Let $(\tau, \kappa) \in X \times X$ be such that $P(\tau, \kappa) = [N]$. Then,*

(a) *The following conditions are equivalent:*

(i) $\ell(\tau) = \ell(\kappa)$, and

(ii) *There is an involution $\lambda \in \Sigma_N$ that conjugates each of τ and κ to the other.*

(b) *If the conditions (i) and (ii) hold, then $N - 2\ell(\tau) \leq 1$. For any $\sigma \in \Sigma_N$ that commutes with both τ and κ , $S(\sigma, \tau)S(\sigma, \kappa) = 1$.*

(c) *If the conditions (i) and (ii) do not hold, there is an involution $\sigma \in \Sigma_N$ commuting with both τ and κ and is such that $S(\sigma, \tau)S(\sigma, \kappa) = -1$.*

Proof. Suppose that there is an element p_1 of $[N]$ that is moved by one of the involutions, say τ , to p_2 and fixed by κ . Then, unless $N = 2$, p_2 is necessarily moved by κ to p_3 for otherwise $H = \{p_1, p_2\}$ would be a non-empty subset of $[N]$ with non-empty complement K so that both τ and κ are in $\Sigma_H \times \Sigma_K$. Then, unless $N = 3$, p_3 is necessarily moved by τ , to p_4 , for otherwise, $H = \{p_1, p_2, p_3\}$ would be a non-empty subset of $[N]$ with non-empty complement K so that both τ and κ are in $\Sigma_H \times \Sigma_K$. Proceeding in this manner, we see that one of the following must hold:

Case I: N is even and there is a renumbering $\{p_1, p_2, \dots, p_{2j}\}$ of $[N]$ with $\tau = (p_1 p_2)(p_3 p_4) \dots (p_{2j-1} p_{2j})$ and $\kappa = (p_2 p_3)(p_4 p_5) \dots (p_{2j-2} p_{2j-1})$, or

Case II: N is odd and there is a renumbering $\{p_1, p_2, \dots, p_{2j+1}\}$ of $[N]$ with $\tau = (p_1 p_2)(p_3 p_4) \dots (p_{2j-1} p_{2j})$ and $\kappa = (p_2 p_3)(p_4 p_5) \dots (p_{2j} p_{2j+1})$.

On the other hand, if no element of $[N]$ is left fixed by either κ or τ , then N is clearly necessarily even and a similar analysis shows that the following holds:

Case III: N is even and there is a renumbering $\{p_1, p_2, \dots, p_{2j}\}$ of $[N]$ with $\tau = (p_1 p_2)(p_3 p_4) \dots (p_{2j-1} p_{2j})$ and $\kappa = (p_2 p_3)(p_4 p_5) \dots (p_{2j-2} p_{2j-1})(p_{2j} p_1)$.

To prove (a), since (ii) clearly implies (i), now suppose that (i) holds. Then we are in either Case II or Case III. In Case II, the involution λ that interchanges p_i with p_{N+1-i} conjugates each of τ and κ to the other while

in Case III, the involution λ that interchanges p_i with p_{N+2-i} works (the addition is mod N).

The first assertion of (b) is clear since in Case II, $N = 2j + 1$ while in Case III, $N = 2j$ with $j = \ell(\tau) = \ell(\kappa)$. The second assertion is a little more delicate. Observe that as $P(\tau, \kappa) = [N]$, any σ commuting with both τ and κ is determined by any one $\sigma(k)$. In Case II, since p_1 is the unique element of $[N]$ moved by τ and fixed by κ , $\sigma(p_1) = p_1$ and hence $\sigma = 1_G$.

In Case III, the elements that commute with both τ and κ form a dihedral group generated by the involution $\sigma_2 = (p_1 p_{2j})(p_2 p_{2j-1}) \cdots (p_j p_{j+1})$ and the element $\sigma_1 = (p_1 p_3 \cdots p_{2j-1})(p_2 p_4 \cdots p_{2j})$. Since S is a character on the centraliser of τ , it suffices to check that $S(\sigma_i, \tau)S(\sigma_i, \kappa) = 1$. Inspection shows that $S(\sigma_1, \kappa) = 1 = S(\sigma_1, \tau)$ and that $S(\sigma_2, \kappa) = (-1)^j = S(\sigma_2, \tau)$, finishing the proof of (b).

The hypothesis of (c) implies that we are in Case I and here again the involution that interchanges p_i with p_{N+1-i} is seen to satisfy the conclusion of (c). \square

Corollary 3 *Let $(\tau_1, \kappa_1), (\tau_2, \kappa_2) \in X \times X$ with $P(\tau_1, \kappa_1) = [N] = P(\tau_2, \kappa_2)$ and with $\ell(\tau_1) = \ell(\kappa_1)$ and $\ell(\tau_2) = \ell(\kappa_2)$. Then there is a $\sigma \in \Sigma_N$ such that $(\tau_2, \kappa_2) = (\sigma\tau_1\sigma^{-1}, \sigma\kappa_1\sigma^{-1})$.*

Proof. It follows from Lemma 2 that for each of (τ_1, κ_1) and (τ_2, κ_2) we are either Case II or Case III, and in the same case for both since that depends only on whether N is odd or even. Thus from the proof of that lemma, (τ_1, κ_1) and (τ_2, κ_2) differ only by a renumbering of $[N]$ which yields the sought after σ . \square

Lemma 4 *Let $(\tau, \kappa) \in X \times X$ with $P(\tau, \kappa) = \{H_1, \dots, H_t\}$. Then exactly one of the following two conditions holds:*

- (i) *There is a $\sigma \in \Sigma_N$ commuting with τ and κ such that $S(\sigma, \tau)S(\sigma, \kappa) = -1$.*
- (ii) *There is an involution $\lambda \in \Sigma_{H_1} \times \cdots \times \Sigma_{H_t}$ such that $\kappa = \lambda\tau\lambda^{-1}$.*

Proof. To see that at least one of the two conditions holds, consider the elements τ_i and κ_i . If their lengths are equal for all $i = 1, \dots, t$, then by Lemma 2, they are conjugate by involutions $\lambda_i \in \Sigma_{H_i}$. The product $\lambda = \lambda_1 \cdots \lambda_t$ gives an involution living in $\Sigma_{H_1} \times \cdots \times \Sigma_{H_t}$ that conjugates τ to κ . Otherwise, for some i there is, by Lemma 2 again, a $\sigma_i \in \Sigma_{H_i}$ that commutes with both τ_i and κ_i and is such that, in Σ_{H_i} , $S(\sigma_i, \tau_i)S(\sigma_i, \kappa_i) = -1$. Simply regard $\sigma = \sigma_i$ as an element of Σ_N to get an element satisfying (i).

To see that they are mutually exclusive, suppose that (ii) holds and consider any $\sigma \in \Sigma_N$ commuting with both τ and κ . Then σ preserves the

characteristic partition in the sense that $\{\sigma(H_1), \dots, \sigma(H_t)\} = \{H_1, \dots, H_t\}$. Recalling the definition of S , a little thought shows that $S(\sigma, \tau)$ is the product of all $S(\sigma_i, \tau_i)$ where the product is over all i for which $\sigma(H_i) = H_i$ and $\sigma_i = \sigma|_{H_i}$. A similar statement holds for $S(\sigma, \kappa)$. We now note that σ_i commutes with τ_i and κ_i and appeal to Lemma 2(b) to finish the proof. \square

We note that a consequence of this lemma is that for any $(\tau, \kappa) \in X \times X$, whether the condition (i) or (ii) of the lemma holds is a property of its G -orbit - a fact that does not seem to be directly verifiable for the condition (i).

3 The main theorem

Proposition 5 *For $j \neq k$ the representations π_j and π_k are mutually disjoint.*

Proof. It is to be seen that any G -map $T : V_j \rightarrow V_k$ is zero. Since the entries of T satisfy the system of equations (2), it suffices to see that for any $\tau \in X_j$ and $\kappa \in X_k$ there exists a $\sigma \in G$ commuting with both so that $S(\sigma, \tau)$ and $S(\sigma, \kappa)$ have different signs. But that follows from Lemma 4 since the case (ii) there cannot occur for involutions of different length. \square

Proposition 6 *For any j , the endomorphism ring $\text{End}_G(V_j)$ is commutative.*

Proof. Let T be an arbitrary element of $\text{End}_G(V_j)$. We assert that the matrix of T with respect to the basis X_j is symmetric. To see this, let τ, τ' be arbitrary elements of X_j . If there is an element $\sigma \in \Sigma_N$ commuting with both so that $S(\sigma, \tau)S(\sigma, \tau') = -1$, then equations (2) imply that both T_τ^τ and $T_{\tau'}^{\tau'}$ are 0. Otherwise, Lemma 4 implies that there is an involution $\lambda \in \Sigma_N$ that conjugates each of τ and τ' to the other. By the basic property of the S function, $1 = S(1_G, \tau) = S(\lambda, \tau)S(\lambda^{-1}, \lambda\tau\lambda^{-1}) = S(\lambda, \tau)S(\lambda, \kappa)$. Now again appeal to the equations (2) to see that $T_\tau^\tau = T_{\tau'}^{\tau'}$ in this case too.

But now, $\text{End}_G(V_j)$ is identified with an algebra of symmetric matrices and is therefore commutative. \square

Proposition 7 *For any j , the dimension of $\text{End}_G(V_j)$ is equal to the number of partitions of N with $N - 2j$ odd parts.*

Proof. It should be clear from the equations (2) that the matrix entries of an arbitrary $T \in \text{End}_G(V_j)$ are completely determined by the values of $T_\tau^{\tau'}$ as (τ, τ') range over representatives of simultaneous conjugation G -orbits in $X_j \times X_j$.

Furthermore, if for a $(\tau, \tau') \in X_j \times X_j$, there exists $\sigma \in G$ commuting with both such that $S(\sigma, \tau)S(\sigma, \tau') = -1$ then the entries of T in the orbit of (τ, τ') vanish as observed earlier. Conversely, suppose that it is a consequence of equations (2) that $T_\tau^{\tau'} = 0$ for all $T \in \text{End}_G(V_j)$. Then necessarily there exist (τ_i, τ'_i) for $i = 1, 2, \dots, n$ in the orbit of (τ, τ') and $\sigma_i \in G$ so that for $i = 1, 2, \dots, n-1$, we have $(\tau_{i+1}, \tau'_{i+1}) = \sigma_i \cdot (\tau_i, \tau'_i)$ and $(\tau_1, \tau'_1) = \sigma_n \cdot (\tau_n, \tau'_n)$ and such that

$$\frac{S(\sigma_n, \tau'_n)S(\sigma_{n-1}, \tau'_{n-1}) \cdots S(\sigma_1, \tau'_1)}{S(\sigma_n, \tau_n)S(\sigma_{n-1}, \tau_{n-1}) \cdots S(\sigma_1, \tau_1)} = -1.$$

But now the product $\sigma = \sigma_n \cdots \sigma_1$ is easily verified - using equation (1) - to commute with both and satisfy $S(\sigma, \tau_1)S(\sigma, \tau'_1) = -1$. Therefore the dimension of $\text{End}_G(V_j)$ is equal to the number of G -orbits, under simultaneous conjugation, in $X_j \times X_j$ of pairs (τ, τ') that do not satisfy (i) of Lemma 4.

Take such a pair with characteristic partition $\{H_1, H_2, \dots, H_t\}$. The proofs of Lemmas 2 and 4 show that restricted to each H_i , both τ and τ' have the same length, say, j_i and further that $|H_i| - 2j_i \leq 1$. Thus the numerical characteristic partition of (τ, τ') is a partition of N with $N - 2j$ odd parts which is clearly an invariant of the orbit of (τ, τ') . Corollary 3 implies that this invariant is complete. In other words, two elements of $X_j \times X_j$ with the same numerical characteristic partition are in the same G -orbit.

Conversely, given a partition of N with $N - 2j$ odd parts, take any corresponding (set) partition, say $\{H_1, \dots, H_t\}$, of $[N]$ and for each H_i of cardinality at least 2, choose involutions $\tau_i, \tau'_i \in \Sigma_{H_i}$ as in Case II or Case III of Lemma 2, according as $\#H_i$ is odd or even. Let $\tau = \tau_1\tau_2 \cdots \tau_t$ and $\tau' = \tau'_1\tau'_2 \cdots \tau'_t$. Then Lemma 4 implies that (τ, τ') does not satisfy (i) and then clearly its numerical partition is the one we started with. \square

We now have all the ingredients in place to finish the proof of the main theorem.

Proof of Theorem 1. It follows from Propositions 5 and 6 that $\text{End}_G(A) = \oplus_j \text{End}_G(V_j)$ and is commutative so that any irreducible representation of G occurs in A at most once. Now Proposition 7 implies that the dimension of $\text{End}_G(A)$ is equal to the number of partitions of N which is the number of conjugacy classes and hence irreducible representations of Σ_N , and so each irreducible representation of Σ_N does indeed feature in A . \square

Remark 8 *This theorem may be regarded as a ‘structural explanation’ of the numerical coincidence arising out of the R-S-K correspondence - see Exercise 6 of §4.3 of [Flt] - that the number of involutions in a symmetric group is equal to the sum of the degrees of its irreducible representations.*

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